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Research article

Numerical solutions for nonlinear Volterra-Fredholm integral equations of the second kind with a phase lag

Gamal A. Mosa[1], **Mohamed A. Abdou**[2] **and Ahmed S. Rahby**[1,*]

[1] Department of Mathematics, Faculty of Science, Benha University, Egypt
[2] Department of Mathematics, Faculty of Education, Alexandria University, Egypt

* **Correspondence:** Email: a.s.rahby@fsc.bu.edu.eg; Tel: +201004756323.

Abstract: This study is focused on the numerical solutions of the nonlinear Volterra-Fredholm integral equations (NV-FIEs) of the second kind, which have several applications in physical mathematics and contact problems. Herein, we develop a new technique that combines the modified Adomian decomposition method and the quadrature (trapezoidal and Weddle) rules that used when the definite integral could be extremely difficult, for approximating the solutions of the NV-FIEs of second kind with a phase lag. Foremost, Picard's method and Banach's fixed point theorem are implemented to discuss the existence and uniqueness of the solution. Furthermore, numerical examples are presented to highlight the proposed method's effectiveness, wherein the results are displayed in group of tables and figures to illustrate the applicability of the theoretical results.

Keywords: nonlinear Volterra-Fredholm integral equation; Picard's method; Banach's fixed point theorem; modified adomian decomposition method; trapezoidal and Weddle's quadrature rules; contact problems; phase lag
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1. Introduction

Integral equations (IEs) form the common core in the foundations of several science and engineering principles. Thus, several computational approaches have been developed to approximate their solutions [1, 5–7, 11, 13, 16, 17, 19, 21, 23, 24, 26, 29, 35].

The approximate solutions of IEs and the error behavior accompanying these solutions have been investigated in abundance. Brunner [10] and Maleknejad and Hadizadeh [22], for instance, employed the collocation methods and the Adomian decomposition method (ADM), respectively, to approach the numerical solution of the nonlinear Volterra-Fredholm integral equations (NVFIEs) of the second kind. Wazwaz [28] demonstrated the use of the modified ADM (MADM) for mixed NVFIEs of the second kind. El-Borai et al. [14] examined the existence and uniqueness of the solution for the NV-FIE of the second kind and discussed the normality and continuity of the integral operator. Aziz [8] investigated new algorithms for the numerical solution of nonlinear FIEs of the second kind using Haar wavelets. Abdou and Elkojok [2] investigated a numerical method for solving two-dimensional mixed nonlinear IEs with respect to time and position and discussed the existence of a unique solution of nonlinear quadratic IEs of the second kind. Furthermore, Abdou and Raad [3] demonstrated the ADM and its new modifications. In addition, numerical schemes are utilized by Xie et al. [31–34] to solve many types of nonlinear systems of fractional integro and partial differential equations. Brezinski and Redivo-Zaglia [9] explored the extrapolation methods for the numerical solution of nonlinear FIEs of the second kind. Katani [20] applied the quadrature methods to study the numerical solutions of FIEs. Ezquerro and Hernández-Verón [15] demonstrated an approach for obtaining the domains of the existence and uniqueness of the solution for FIEs, including the numerical solutions and assigning priori and posteriori error estimates for these approximations.

The phase lag is extremely important in real-life applications of IEs. Currently, there are single, dual, and three phases, with each phase tied to different applications [4, 12]. In this study, we develop a new technique that combines the MADM and quadrature rules for describing the solution behavior of NV-FIE with the phase lag parameter.

Assuming an NV-FIE of the second kind,

$$\mu\phi(x,t) = f(x,t) + \lambda \int_0^t \int_a^b F(t,\tau)K(x,y)G(y,\tau,\phi(y,\tau))dyd\tau, \tag{1.1}$$

tied with an initial condition

$$\phi(x,0) = \frac{f(x,0)}{\mu} = \varphi(x),$$

where $x = x(x_1, x_2, \ldots, x_n), y = y(y_1, y_2, \ldots, y_n)$, and both $\mu \neq 0$ and $\lambda \neq 0$ are constans. Eq (1.1) will be discussed in the space $L_2[a,b] \times C[0,T]$, $T < 1$, where $[a,b]$ is the domain of integration with respect to position while the time $t \in [0,T]$. Here, the Fredholm integral term is considered in the space $L_2[a,b]$ and the Volterra term is considered in the class $C[0,T]$. Moreover, Eq (1.1) possesses a unique solution under the following conditions:

1. The kernel of position $K(x,y)$ is continuous in $L_2[a,b]$ and satisfies $|K(x,y)| \leq A_1$, whereas the kernel of time $F(t,\tau)$ is continuous in $C[0,T]$ and satisfies $|F(t,\tau)| \leq B_1$, $\forall t,\tau \in [0,T]$, and $0 \leq \tau \leq t \leq T < 1$.
2. The given function $f(x,t)$ is continuous in the space $L_2[a,b] \times C[0,T]$, and its norm is defined as $\|f\|_{L_2[a,b]\times C[0,T]} = \max_{0\leq t\leq T} \left| \int_0^t \left(\int_a^b f^2(x,\tau)dx \right)^{\frac{1}{2}} d\tau \right| \leq C_1$, whereas the unknown function $\phi(x,t)$ exhibits the same behavior as the given function with $\|\phi\| \leq C_2$.
3. (a) The known continuous function $G(x,t,\phi(x,t))$ satisfies the Lipschitz condition $|G(x,t,\phi_2(x,t)) - G(x,t,\phi_1(x,t))| \leq N(x,t)\,|\phi_2(x,t) - \phi_1(x,t)|$, where $\|N\| = \max_{0\leq t\leq T} \left| \int_0^t \left(\int_a^b N^2(x,\tau)dx \right)^{\frac{1}{2}} d\tau \right| \leq D_1$.
   (b) Furthermore, the function $G(x,t,\phi(x,t))$ satisfies the inequality $\max_{0\leq t\leq T} \left| \int_0^t \left\{ \int_a^b |G(x,\tau,\phi(x,\tau)))|^2 dx \right\}^{\frac{1}{2}} d\tau \right| \leq D_2\|\phi\|$,

where A_1, B_1, C_1, C_2, D_1, and D_2 are positive constants.

Theorem 1. ***(without proof)*** *[14] If conditions (1), (2), and (3.a) are satisfied, then Eq (1.1) has a unique solution $\phi(x,t)$ in the space $L_2[a,b] \times C[0,T]$, $0 \le t \le T < 1$, under the condition*

$$|\lambda| < \frac{|\mu|}{A_1 B_1 D_1 T}. \tag{1.2}$$

A shock wave [18, 25, 27, 36] is a strong pressure wave in an elastic medium, such as air, water, or a solid substance, produced by any phenomenon that drastically changes the pressure. Shock waves differ from sound waves in that the wavefront where compression occurs is a region of violent and sudden changes in the stress, density, and temperature. Therefore, shock waves travel faster than sound, and their speed increases as the amplitude is raised; however, the intensity of a shock wave decreases faster than that of a sound wave because some of its energy is expended to heat the medium in which it travels. Moreover, shock waves change the electrical, mechanical, and thermal properties of solids; therefore, they can be used to study the equation of the state of any material. The following NV-FIE of the second kind with a phase lag is obtained after the shock wave:

$$\mu\phi(x,t+q) = f(x,t+q) + \lambda \int_0^{t+q} \int_a^b F(t+q,\tau) K(x,y) G(y,\tau,\phi(y,\tau)) dy d\tau, \quad 0 < q << 1. \tag{1.3}$$

This study aims to discuss the stability of the solution for Eq (1.3), which has many physical implications in the fields of engineering, mathematical physics, and biology [13, 19, 29].

2. Mixed integral equation

Applying Taylor's expansion formula and ignoring the second derivatives in Eq (1.3) results in

$$\begin{aligned} \mu\phi(x,t) + q\mu \frac{\partial \phi(x,t)}{\partial t} &= f(x,t) + q\frac{\partial f(x,t)}{\partial t} \\ &+ \lambda \int_0^{t+q} \int_a^b \left(F(t,\tau) + q\frac{\partial F(t,\tau)}{\partial t} \right) K(x,y) G(y,\tau,\phi(y,\tau)) dy d\tau. \end{aligned} \tag{2.1}$$

Accordingly, integrating Eq (2.1) w.r.t. time t under the initial condition yields

$$\begin{aligned} \phi(x,t) &= H(x,t) - \frac{1}{q} \int_0^t \phi(x,z) dz \\ &+ \frac{\lambda}{q\mu} \int_0^t \int_0^{z+q} \int_a^b \left(F(z,\tau) + qP(z,\tau) \right) K(x,y) G(y,\tau,\phi(y,\tau)) dy d\tau dz, \end{aligned} \tag{2.2}$$

where

$$H(x,t) = \varphi(x) + \frac{1}{\mu} \left(f(x,t) - f(x,0) \right) + \frac{1}{q\mu} \int_0^t f(x,z) dz$$

$$\text{and } P(z,\tau) = \frac{\partial F(z,\tau)}{\partial z}.$$

By interchanging the integral in the plan τz and the plan zy, Eq (2.3) can be expressed as

$$\begin{aligned}\phi(x,t) &= H(x,t) - \frac{1}{q}\int_0^t \phi(x,z)dz \\ &+ \frac{\lambda}{q\mu}\int_0^q \int_a^b \Theta(t,\tau)K(x,y)G(y,\tau,\phi(y,\tau))dyd\tau \\ &+ \frac{\lambda}{q\mu}\int_q^{t+q} \int_a^b \Psi(t,\tau)K(x,y)G(y,\tau,\phi(y,\tau))dyd\tau,\end{aligned} \tag{2.3}$$

where

$$\Theta(t,\tau) = \int_0^t (F(z,\tau) + qP(z,\tau))\,dz \text{ and } \Psi(t,\tau) = \int_{\tau-q}^t (F(z,\tau) + qP(z,\tau))\,dz.$$

Applying conditions (1) and (2), the following statements can be generalized:

$\acute{1}$. The continuous functions $\Theta(t,\tau)$ and $\Psi(t,\tau) \in C[0,T]$ satisfy the conditions $|\Theta(t,\tau)| \leq B_2$ and $|\Psi(t,\tau)| \leq B_3,\ \forall\ t,\tau \in [0,T], and\ 0 \leq \tau \leq t \leq T < 1$.

$\acute{2}$. The given function $H(x,t)$ is continuous in $L_2[a,b] \times C[0,T]$, and its norm is defined as $\|H(x,t)\| = \max\limits_{0<t\leq T} \int_0^t \left[\int_a^b H^2(x,\tau)dx\right]^{\frac{1}{2}} d\tau \leq C_3$,

where $B_2,\ B_3$, and C_3 are positive constants.

2.1. Stability of the solution for the mixed integral equation

This section discusses the stability of the solution for the nonlinear mixed IE represented in Eq (2.3).

Theorem 2. *If conditions ($\acute{1}$), ($\acute{2}$), and (3.a) are satisfied, then Eq (2.3) has a unique and stable solution $\phi(x,t)$ in the space $L_2[a,b] \times C[0,T]$, $0 \leq t \leq T < 1$, under the condition*

$$|\lambda| < \frac{|\mu|(T-q)}{A_1 D_1 (qB_2 + TB_3)}. \tag{2.4}$$

Proof. Applying Picard's method, a solution for Eq (2.3) can be constructed as a sequence of functions $\{\phi_n(x,t)\}$ as $n \to \infty$; thus,

$$\phi(x,t) = \lim_{n\to\infty} \phi_n(x,t), \tag{2.5}$$

where

$$\phi_n(x,t) = \sum_{i=0}^n u_i(x,t), \quad n = 1,2,3,... \tag{2.6}$$

and the functions $u_n(x,t), n = 0,1,2,...$, are continuous functions defined as

$$u_n(x,t) = \phi_n(x,t) - \phi_{n-1}(x,t), \text{ and } u_0(x,t) = H(x,t). \tag{2.7}$$

Lemma 1. *If the series $\sum\limits_{i=0}^n u_i(x,t)$ is uniformly convergent, then $\phi(x,t)$ represents a solution of Eq (2.3).*

Proof. We construct a sequence $\phi_m(x,t)$ based on

$$\begin{aligned}\phi_m(x,t) &= H(x,t) - \frac{1}{q}\int_0^t \phi_{m-1}(x,z)dz \\ &+ \frac{\lambda}{q\mu}\int_0^q\int_a^b \Theta(t,\tau)K(x,y)G(y,\tau,\phi_{m-1}(y,\tau))dyd\tau \\ &+ \frac{\lambda}{q\mu}\int_q^{t+q}\int_a^b \Psi(t,\tau)K(x,y)G(y,\tau,\phi_{m-1}(y,\tau))dyd\tau.\end{aligned} \tag{2.8}$$

Employing Eq (2.8) to Eq (2.7) using the norm properties,

$$\begin{aligned}\|u_m(x,t)\| &\le \frac{1}{q}\|u_{m-1}(x,z)\|\|\int_0^t dz\| \\ &+ \|\frac{\lambda}{q|\mu|}\int_0^q\int_a^b |\Theta(t,\tau)||K(x,y)|N(y,\tau)||u_{m-1}(y,\tau)dyd\tau\| \\ &+ \|\frac{\lambda}{q|\mu|}\int_q^{t+q}\int_a^b |\Psi(t,\tau)||K(x,y)|N(y,\tau)||u_{m-1}(y,\tau)dyd\tau\|.\end{aligned} \tag{2.9}$$

Subsequently, the mathematical induction and conditions (1), (3.a), ($\acute{1}$), and ($\acute{2}$) are applied to obtain

$$\|u_m(x,t)\| \le \eta_1^m C_3, \quad \eta_1 = \frac{T}{q} + |\frac{\lambda}{q\mu}|A_1D_1(qB_2 + TB_3) < 1. \tag{2.10}$$

Therefore,

$$|\lambda| < \frac{|\mu|(T-q)}{A_1D_1(qB_2+TB_3)}, \tag{2.11}$$

which implies the sequence $\phi_n(x,t)$ has a convergent solution. Thus, for $n \to \infty$, $\phi_n(x,t) = \sum_{i=0}^{n} u_i(x,t)$ represents a solution of Eq (2.3). □

Lemma 2. *The function $\phi(x,t)$ of the series (2.6) represents a unique solution of Eq (2.3).*

Proof. Suppose there exists another continuous solution $\tilde{\phi}(x,t)$ of Eq (2.3), then

$$\begin{aligned}\|\phi(x,t) - \tilde{\phi}(x,t)\| &\le \| - \frac{1}{q}\int_0^t (\phi(x,z) - \tilde{\phi}(x,z))dz\| \\ &+ \|\frac{\lambda}{q\mu}\int_0^q\int_a^b \Theta(t,\tau)K(x,y)(G(y,\tau,\phi(y,\tau)) - G(y,\tau,\tilde{\phi}(y,\tau)))dyd\tau\,\| \\ &+ \|\frac{\lambda}{q\mu}\int_q^{t+q}\int_a^b \Psi(t,\tau)K(x,y)(G(y,\tau,\phi(y,\tau)) - G(y,\tau,\tilde{\phi}(y,\tau)))dyd\tau\|.\end{aligned} \tag{2.12}$$

Note that under the given conditions, inequality (2.12) yields

$$\|\phi(x,t) - \tilde{\phi}(x,t)\| \le \eta_1\|\phi(x,t) - \tilde{\phi}(x,t)\|. \tag{2.13}$$

As $\eta_1 < 1$, it is implied that $\phi(x,t) = \tilde{\phi}(x,t)$. □

□

2.2. *Normality and continuity of an integral operator*

To prove the normality and continuity of the reduced mixed IE (2.3), it will be first expressed in its integral operator form

$$\overline{W}\phi = H(x,t) + W\phi \text{ and } \overline{W}\phi = \mu\phi \tag{2.14}$$

where

$$W\phi = -W_1\phi + W_2\phi + W_3\phi, \quad W_1\phi = \frac{1}{q}\int_0^t \phi(x,z)dz,$$

$$W_2\phi = \frac{\lambda}{q}\int_0^q \int_a^b \Theta(t,\tau)K(x,y)G(y,\tau,\phi(y,\tau))dyd\tau,$$

and

$$W_3\phi = \frac{\lambda}{q}\int_q^{t+q} \int_a^b \Psi(t,\tau)K(x,y)G(y,\tau,\phi(y,\tau))dyd\tau.$$

2.2.1. Normality of the integral operator

Form the norm properties,

$$\begin{aligned} \|W\phi\| \leq \| &- \frac{1}{q}\int_0^t \phi(x,z)dz\| \\ &+ \|\frac{\lambda}{q}\int_0^q \int_a^b \Theta(t,\tau)\, K(x,y)G(y,\tau,\phi(y,\tau))dy\, d\tau\| \\ &+ \|\frac{\lambda}{q}\int_q^{t+q} \int_a^b \Psi(t,\tau)K(x,y)G(y,\tau,\phi(y,\tau))dy\, d\tau\|. \end{aligned} \tag{2.15}$$

Using the norm properties in $L_2[a,b], C[0,T]$ with the conditions (1), ($\acute{1}$), and (3.b), inequality (2.15) can be expressed as

$$\|W\phi\| \leq \eta_2\|\phi\|, \quad \eta_2 = \frac{T}{q} + |\frac{\lambda}{q}|(qB_2 + TB_3)A_1D_2 < 1, \tag{2.16}$$

to obtain

$$|\lambda| < \frac{T-q}{(qB_2 + TB_3)\, A_1 D_2}.$$

Therefore, the integral operator W has a normality that leads directly, after using condition ($\acute{2}$), to the normality of the operator $\overline{W}$.

2.2.2. Continuity of the integral operator

Assume two potential functions $\phi_1(x,t)$ and $\phi_2(x,t)$ in the space $L_2[a,b] \times C[0,T]$. Applying the conditions,

$$\|W\phi_1 - W\phi_2\| \leq \eta_3\|\phi_1 - \phi_2\|, \quad \eta_3 < 1, \tag{2.17}$$

and thus,

$$|\lambda| < \frac{T-q}{(qB_2 + TB_3)\, A_1 D_1}.$$

Inequality (2.17) leads to the continuity of the integral operator W. Furthermore, W is a contraction operator in the space $L_2[a,b] \times C[0,T]$. According to the Banach's fixed point theorem, W contains a unique fixed point. If the normality and continuity of the integral operator is employed, then the existence and uniqueness of the reduced mixed IE (2.3) are approved.

3. Modified Adomian decomposition method for NV-FIEs

Several numerical techniques can be applied to solve the NV-FIEs of the second kind [19, 28–30]. However, herein we seek to develop a new approach that combines MADM and quadrature rules. Therefore, the solution of Eq (2.3) can be expressed as

$$\phi(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{3.1}$$

and its approximate solution can be expressed as

$$\phi_N(x,t) = \sum_{n=0}^{N} u_n(x,t). \tag{3.2}$$

Accordingly, the nonlinear term of Eq (2.3) can be decomposed into an infinite series of Adomian polynomials as

$$G(y,\tau,\phi(y,\tau)) = \sum_{n=0}^{\infty} A_n(y,\tau), \tag{3.3}$$

where the traditional formula of $A_n(y,\tau)$ is

$$A_n(y,\tau) = \frac{1}{n!}\left(\frac{d^n}{d\eta^n} G\left(y,\tau,\sum_{l=0}^{\infty} \eta^l u_l(y,\tau)\right)\right)_{\eta=0},$$

and the free term can be modified into the form

$$H(x,t) = \sum_{n=0}^{\infty} H_n(x,t). \tag{3.4}$$

Consequently,

$$\begin{aligned} u_0(x,t) &= H_0(x,t) \\ u_n(x,t) &= H_n(x,t) - \tfrac{1}{q}\int_0^t u_{n-1}(x,z)dz \\ &+ \tfrac{\lambda}{q\mu}\int_0^q \int_a^b \Theta(t,\tau)\, K(x,y) A_{n-1}(y,\tau) dy\, d\tau \\ &+ \tfrac{\lambda}{q\mu}\int_q^{t+q} \int_a^b \Psi(t,\tau) K(x,y) A_{n-1}(y,\tau) dy\, d\tau,\ n = 1,2,...,N. \end{aligned} \tag{3.5}$$

To obtain a more accurate solution for the definite integral when it could be extremely difficult in (3.5), any of these quadrature rules can be applied instead.

3.1. Trapezoidal rule (TR)

Suppose the interval $[a,b]$ is divided into M subintervals of equal width $h_t = \frac{b-a}{M}$. Then, using the equally spaced sample points $x_k = h_t k + x_0$, $k = 0,1,2,...,M$, yields

$$\begin{aligned}\mathcal{T}_{n-1}(x,\tau) &= \int_a^b K(x,y)A_{n-1}(y,\tau)dy \\ &= \frac{h_t}{2}\left(K(x,y_0)A_{n-1}(y_0,\tau) + K(x,y_M)A_{n-1}(y_M,\tau)\right) \\ &+ h_t \sum_{k=1}^{M-1} K(x,y_k)A_{n-1}(y_k,\tau).\end{aligned} \tag{3.6}$$

Hence, (3.5) becomes

$$\begin{aligned} u_0(x,t) &= H_0(x,t) \\ u_n(x,t) &= H_n(x,t) - \tfrac{1}{q}\int_0^t u_{n-1}(x,z)dz \\ &+ \tfrac{\lambda}{q\mu}\int_0^q \Theta(t,\tau)\,\mathcal{T}_{n-1}(x,\tau)\,d\tau \\ &+ \tfrac{\lambda}{q\mu}\int_q^{t+q} \Psi(t,\tau)\mathcal{T}_{n-1}(x,\tau)\,d\tau,\ n = 1,2,...,N.\end{aligned} \tag{3.7}$$

3.2. Weddle's rule (WR)

If the interval $[a,b]$ is divided into $6m$ subintervals of equal width $h_w = \frac{b-a}{6m}$, then applying the equally spaced sample points $y_k = h_w k + y_0$, $k = 0,1,2,...,6m$, yields

$$\begin{aligned}\mathcal{W}_{n-1}(x,\tau) &= \int_a^b K(x,y)A_{n-1}(y,\tau)dy \\ &= \frac{3h_w}{10}\sum_{k=1}^{m} K(x,y_{6k-6})A_{n-1}(y_{6k-6},\tau) \\ &+ \frac{3h_w}{10}\sum_{k=1}^{m} 5K(x,y_{6k-5})A_{n-1}(x_{6k-5},\tau) \\ &+ \frac{3h_w}{10}\sum_{k=1}^{m}\left(K(x,y_{6k-4})A_{n-1}(y_{6k-4},\tau) + 6K(x,y_{6k-3})A_{n-1}(y_{6k-3},\tau)\right) \\ &+ \frac{3h_w}{10}\sum_{k=1}^{m}\left(K(x,y_{6k-2})A_{n-1}(y_{6k-2},\tau)\right) \\ &+ \frac{3h_w}{10}\sum_{k=1}^{m}\left(5K(x,y_{6k-1})A_{n-1}(y_{6k-1},\tau) + K(x,y_{6k}A_{n-1}(y_{6k},\tau)\right).\end{aligned} \tag{3.8}$$

So, (3.5) becomes

$$\begin{aligned} u_0(x,t) &= H_0(x,t) \\ u_n(x,t) &= H_n(x,t) - \tfrac{1}{q}\int_0^t u_{n-1}(x,z)dz \\ &+ \tfrac{\lambda}{q\mu}\int_0^q \Theta(t,\tau)\,)\mathcal{W}_{n-1}(x,\tau)\,d\tau \\ &+ \tfrac{\lambda}{q\mu}\int_q^{t+q} \Psi(t,\tau)\mathcal{W}_{n-1}(x,\tau)\,d\tau,\ n = 1,2,...,N.\end{aligned} \tag{3.9}$$

3.3. Convergence analysis

Now, we shall present the sufficient condition for convergence of considered series.

Theorem 3. *If their exists constants $\alpha \in (0,1)$ and $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$, the following inequality*

$$\|u_{k+1}\| \leq \alpha \|u_k\|. \tag{3.10}$$

is satisfied, then the series solution (3.1) of Eq (2.3) is uniformly convergent in $\mathcal{I} = [a,b] \times [0,T]$.

Proof. Denoting $\mathbb{E} = (C[\mathcal{I}], \|\cdot\|)$ is the Banach space of all continuous functions on $\mathcal{I}$ with the norm $\|\phi(x,t)\| = \max_{\forall x,t\in\mathcal{I}} |\phi(x,t)|$. Let, ϕ_n and ϕ_m be arbitrary partial sums with $n \geq m$. We are going to prove that $\{\phi_n\}$ is a Cauchy sequence in $\mathbb{E}$, so we estimate the following norm

$$\|\phi_{n+1} - \phi_n\| = \|u_{n+1}\| \leq \alpha\|u_n\| \leq \alpha^2\|u_{n-1}\| \leq ... \leq \alpha^{n-k_0+1}\|u_{k_0}\| \tag{3.11}$$

Now for any $n, k \in \mathbb{N}, n \geq k \geq k_0$, we have

$$\begin{aligned} \|\phi_n - \phi_k\| &\leq \|\phi_n - \phi_{n-1}\| + \ldots + \|\phi_{k+1} - \phi_k\| \\ &\leq \alpha^{n-k_0} \left\|u_{k_0}\right\| + \ldots + \alpha^{k+1-k_0} \left\|u_{k_0}\right\| \\ &= \alpha^{k+1-k_0} \frac{1-\alpha^{n-k}}{1-\alpha} \left\|u_{k_0}\right\|. \end{aligned} \tag{3.12}$$

Since $\alpha \in (0,1)$, therefore it implies that $1 - \alpha^{n-k} \leq 1$ and

$$\|\phi_n - \phi_k\| \leq \frac{\alpha^{k+1-k_0}}{1-\alpha} \left\|u_{k_0}\right\|. \tag{3.13}$$

So, $\|\phi_n - \phi_k\| \to 0$ as $k \to \infty$, therefore it implies $\{\phi_n\}$ is a Cauchy sequence in $\mathbb{E}$ and we can deduce that the series $\sum_{i=0}^{\infty} u_i(x,t)$ is convergent. □

3.4. Error estimate

Next theorem concerns the estimation of error of the approximate solution $\phi_N(x,t)$.

Theorem 4. *If assumptions of Theorem 3 are satisfied, $N \in \mathbb{N}$ and $N \geq k_0$, then we obtain the estimation of error of the approximate solution such that*

$$\|\phi(x,t) - \phi_N(x,t)\| \leq \frac{\alpha^{N+1-k_0}}{1-\alpha} \left\|u_{k_0}\right\|. \tag{3.14}$$

Proof. Let $N \in \mathbb{N}$ and $N \geq k_0$, we get

$$\begin{aligned}
\|\phi(x,t) - \phi_N(x,t)\| &= \sup_{(x,t)\in\mathcal{I}} \left| \phi(x,t) - \sum_{n=0}^{N} u_n(x,t) \right| \\
&\leq \sup_{(x,t)\in\mathcal{I}} \left(\sum_{n=N+1}^{\infty} |u_n(x,t)| \right) \\
&\leq \sum_{n=N+1}^{\infty} \sup_{(x,t)\in\mathcal{I}} (|u_n(x,t)|) \\
&\leq \sum_{n=N+1}^{\infty} \alpha^{n-k_0} \left\| u_{k_0} \right\| \\
&= \frac{\alpha^{N+1-k_0}}{1-\alpha} \left\| u_{k_0} \right\|.
\end{aligned} \tag{3.15}$$

In particular case, at $k_0 = 0$, we get

$$\|\phi(x,t) - \phi_N(x,t)\| \leq \frac{\alpha^{N+1}}{1-\alpha} \|u_0\|. \tag{3.16}$$

□

4. Numerical results and discussion

In this section, the methods presented above will be utilized in some applications to explain the behavior of the solution error for some NV-FIEs of the second kind.

Application 1. *Consider the NV-FIE of the second kind,*

$$\phi(x, t+0.0001) = f(x, t+0.0001) + \int_0^{t+0.0001} \int_0^1 (t+0.0001)\tau^2 x y^2 \phi^2(y,\tau)\, dy\, d\tau, \tag{4.1}$$

$$\text{“ } \phi(x,t) = t \ln(1+x) \text{”.}$$

For this application, Table 1 presents the absolute values of error using MADM, MADMTR, and MADMWR for Eq (4.1) in the interval $x \in [0,1]$*, using different values of* $t_i \in [0, 0.6]$, $i = 0, 1, 2$ *with* $N = 3$*. Here, the results were plotted in a group of Figures 1–3 to display the error behavior for each method. In addition, Table 2 lists the maximum error* $E_{max}(t) = \max_i |\phi(x_i,t) - \phi_N(x_i,t)| \ \forall\ x_i \in [0,1]$ *for some* $t \in [0, 0.6]$.

Table 1. Absolute error of the solution of Eq (4.1) using the previously presented methods at $0 \leq T \leq 0.6$.

	$t_0 = 0$			$t_1 = 0.3$			$t_2 = 0.6$		
x_i	MADM	MADM-TR	MADM-WR	MADM	MADM-TR	MADM-WR	MADM	MADM-TR	MADM-WR
0	0	0	0	0	0	0	0	0	0
0.1	0	4.59E-48	2.27E-56	3.68E-11	6E-08	3.68E-11	1.88E-08	7.68E-06	1.88E-08
0.2	0	9.18E-48	4.53E-56	7.37E-11	1.2E-07	7.37E-11	3.77E-08	1.54E-05	3.76E-08
0.3	0	1.38E-47	6.8E-56	1.1E-10	1.8E-07	1.1E-10	5.65E-08	2.31E-05	5.65E-08
0.4	0	1.84E-47	9.06E-56	1.47E-10	2.4E-07	1.47E-10	7.54E-08	3.07E-05	7.53E-08
0.5	0	2.29E-47	1.13E-55	1.84E-10	3E-07	1.84E-10	9.42E-08	3.84E-05	9.41E-08
0.6	0	2.75E-47	1.36E-55	2.21E-10	3.6E-07	2.21E-10	1.13E-07	4.61E-05	1.13E-07
0.7	0	3.21E-47	1.59E-55	2.58E-10	4.2E-07	2.58E-10	1.32E-07	5.38E-05	1.32E-07
0.8	0	3.67E-47	1.81E-55	2.95E-10	4.8E-07	2.95E-10	1.51E-07	6.15E-05	1.51E-07
0.9	0	4.13E-47	2.04E-55	3.31E-10	5.4E-07	3.31E-10	1.7E-07	6.92E-05	1.69E-07
1	0	4.59E-47	2.27E-55	3.68E-10	6E-07	3.68E-10	1.88E-07	7.68E-05	1.88E-07

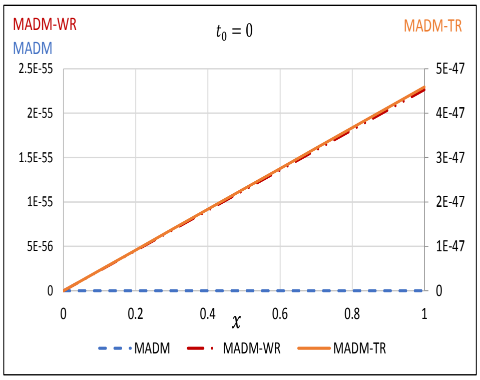

Figure 1. Comparison of the errors obtained using the previously presented methods at $t = 0$ for Eq (4.1).

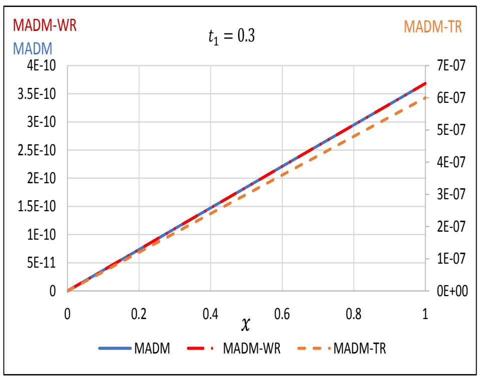

Figure 2. Comparison of the errors obtained using the previously presented methods at $t = 0.3$ for Eq (4.1).

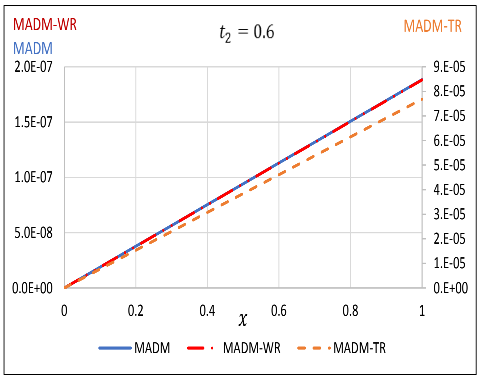

Figure 3. Comparison of the errors obtained using the previously presented methods at $t = 0.6$ for Eq (4.1).

Table 2. The maximum error $E_{max}(t)$ for different values of t for Eq (4.1).

t	MADM	MADM-TR	MADM-WR
0	0	4.59×10^{-47}	2.27×10^{-55}
0.3	3.68×10^{-10}	5.99×10^{-7}	3.68×10^{-10}
0.6	1.88×10^{-7}	7.68×10^{-5}	1.88×10^{-7}

Application 2. *Consider the NV-FIE of the second kind,*

$$\phi(x, t + 0.003) = f(x, t + 0.0003) + \int_0^{t+0.0003} \int_0^{\pi} (t+0.0003)\tau \cos x \ \sin y \ \phi^2(y, \tau) \ dy, \ d\tau, \quad (4.2)$$
$$\text{“}\phi(x, t) = t(\sin x + \cos x)\text{”}.$$

Table 3 lists the absolute error values obtained using MADM, MADM-TR and MADM-WR for Eq (4.2) in the interval $x \in [0, \pi]$ at different values of $t_i \in [0, 0.4]$, $i = 0, 1, 2$ with $N = 3$. Figures 4–6 show the graphically display the results that can be used to investigate the error behavior for each method. Moreover, Table 4 indicates the maximum error $E_{max}(t)$ $\forall$ $x_i \in [0, \pi]$ for some $t \in [0, 0.4]$.

Table 3. Absolute error of the solution of Eq (4.2) using the previously presented methods at $0 \leq T \leq 0.4$.

	$t_0 = 0$			$t_1 = 0.2$			$t_2 = 0.4$		
x_i	MADM	MADM-TR	MADM-WR	MADM	MADM-TR	MADM-WR	MADM	MADM-TR	MADM-WR
0	0	4.39E-39	2.94E-47	5.41E-11	1.02E-06	5.41E-11	1.38E-08	6.53E-05	1.38E-08
$\frac{\pi}{10}$	0	4.17E-39	2.8E-47	5.14E-11	9.74E-07	5.14E-11	1.31E-08	6.21E-05	1.31E-08
$\frac{\pi}{5}$	0	3.55E-39	2.38E-47	4.37E-11	8.29E-07	4.37E-11	1.12E-08	5.28E-05	1.12E-08
$\frac{3\pi}{10}$	0	2.58E-39	1.73E-47	3.18E-11	6.02E-07	3.18E-11	8.11E-09	3.84E-05	8.11E-09
$\frac{2\pi}{5}$	0	1.36E-39	9.1E-48	1.67E-11	3.17E-07	1.67E-11	4.26E-09	2.02E-05	4.26E-09
$\frac{\pi}{2}$	0	0	0	0	0	0	0	0	0
$\frac{3\pi}{5}$	0	1.36E-39	9.1E-48	1.67E-11	3.17E-07	1.67E-11	4.26E-09	2.02E-05	4.26E-09
$\frac{7\pi}{10}$	0	2.58E-39	1.73E-47	3.18E-11	6.02E-07	3.18E-11	8.11E-09	3.84E-05	8.11E-09
$\frac{4\pi}{5}$	0	3.55E-39	2.38E-47	4.37E-11	8.29E-07	4.37E-11	1.12E-08	5.28E-05	1.12E-08
$\frac{9\pi}{10}$	0	4.17E-39	2.8E-47	5.14E-11	9.74E-07	5.14E-11	1.31E-08	6.21E-05	1.31E-08
π	0	4.39E-39	2.94E-47	5.41E-11	1.02E-06	5.41E-11	1.38E-08	6.53E-05	1.38E-08

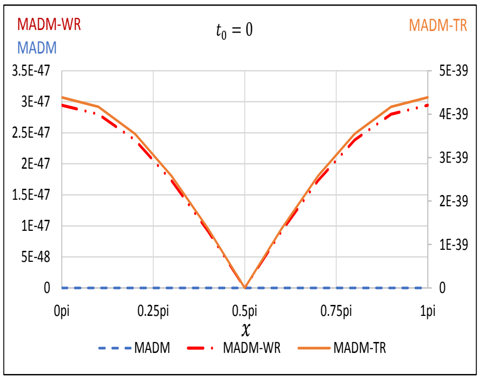

Figure 4. Comparison of the errors obtained using the previously presented methods at $t = 0$ for Eq (4.2).

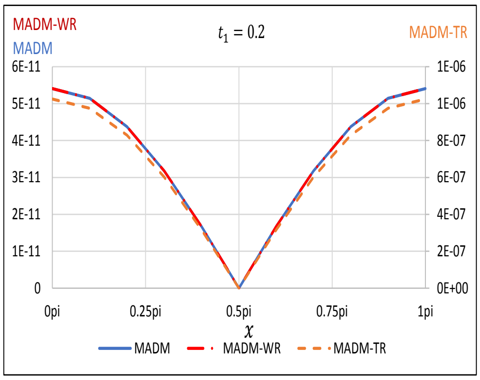

Figure 5. Comparison of the errors obtained using the previously presented methods at $t = 0.2$ for Eq (4.2).

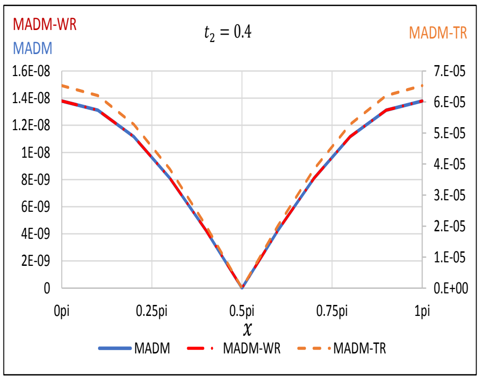

Figure 6. Comparison of the errors obtained using the previously presented methods at $t = 0.4$ for Eq (4.2).

Table 4. The maximum error $E_{max}(t)$ for different values of t for Eq (4.2).

t	MADM	MADM-TR	MADM-WR
0	0	4.39×10^{-39}	2.94×10^{-47}
0.2	5.41×10^{-11}	1.02×10^{-6}	5.41×10^{-11}
0.4	1.38×10^{-8}	6.53×10^{-5}	1.38×10^{-8}

Application 3. *Consider the NV-FIE of the second kind,*

$$\phi(x, t + 0.0002) = f(x, t + 0.0002) + \int_0^{t+0.0002} \int_0^1 (t + 0.0002)^2 \tau^2 x^2 e^y \phi^{\frac{1}{2}}(y, \tau)\, dy\, d\tau, \tag{4.3}$$

“$\phi(x, t) = te^{-x}$”.

Table 5 can be used to investigate the absolute value of the errors obtained using MADM, MADMTR, and MADMWR for Eq (4.3) in the interval $x \in [0, 1]$. *The error behavior for each method at different values of* $t_i \in [0, 0.2]$, $i = 0, 1, 2$ *with* $N = 3$ *is displayed in Figues 7–9. Furthermore, Table 6 shows the maximum error* $E_{max}(t)\ \forall\ x_i \in [0, 1]$ *for some* $t \in [0, 0.2]$.

Table 5. Absolute error of the solution of Eq (4.3) using the previously presented methods at $0 \leq T \leq 0.2$.

	$t_0 = 0$		$t_1 = 0.1$		$t_2 = 0.2$	
x_i	MADM-TR	MADM-WR	MADM-TR	MADM-WR	MADM-TR	MADM-WR
0	0	0	1.13371×10^{-15}	1.42×10^{-15}	2.83×10^{-15}	2.83×10^{-15}
0.1	1.94×10^{-40}	1.94×10^{-40}	1.32×10^{-12}	5.55×10^{-17}	1.19×10^{-10}	4.31×10^{-15}
0.2	7.77×10^{-40}	7.77×10^{-40}	5.28×10^{-12}	5.29×10^{-15}	4.75×10^{-10}	2.22×10^{-14}
0.3	1.75×10^{-39}	1.75×10^{-39}	1.19×10^{-11}	5.38×10^{-16}	1.07×10^{-09}	3.64×10^{-15}
0.4	3.11×10^{-39}	3.11×10^{-39}	2.11×10^{-11}	1.51×10^{-15}	1.91×10^{-09}	8.05×10^{-16}
0.5	4.86×10^{-17}	4.86×10^{-17}	3.29×10^{-11}	1.01×10^{-15}	2.97×10^{-09}	3.48×10^{-15}
0.6	6.99×10^{-17}	6.99×10^{-17}	4.76×10^{-11}	1.11×10^{-15}	4.28×10^{-09}	8.19×10^{-16}
0.7	9.52×10^{-16}	9.52×10^{-16}	6.48×10^{-11}	1.44×10^{-15}	5.82×10^{-09}	1.01×10^{-14}
0.8	1.24×10^{-38}	1.24×10^{-38}	8.46×10^{-11}	5.13×10^{-16}	7.61×10^{-09}	9.05×10^{-15}
0.9	1.57×10^{-38}	1.57×10^{-38}	1.07×10^{-10}	1.47×10^{-15}	9.63×10^{-09}	1.05×10^{-14}
1	1.94×10^{-38}	1.94×10^{-38}	1.32×10^{-10}	2.82×10^{-15}	1.19×10^{-08}	7.97×10^{-15}

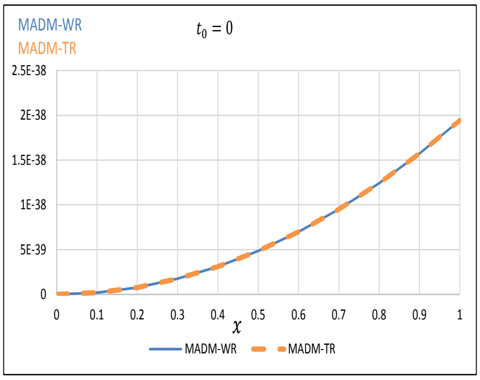

Figure 7. Comparison of the errors obtained using the previously presented methods at $t = 0$ for Eq (4.3).

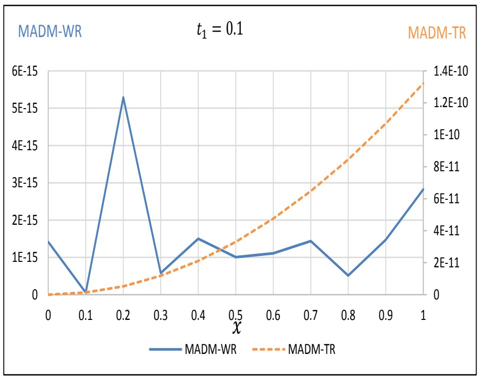

Figure 8. Comparison of the errors obtained using the previously presented methods at $t = 0.1$ for Eq (4.3).

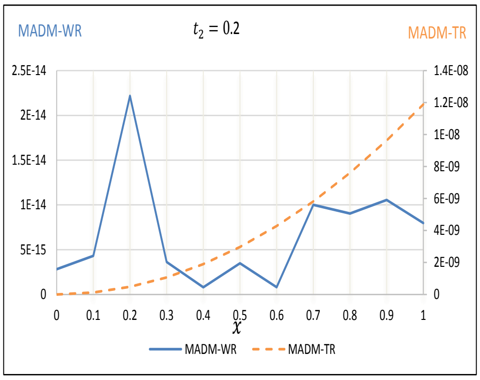

Figure 9. Comparison of the errors obtained using the previously presented methods at $t = 0.2$ for Eq (4.3).

Table 6. The maximum error $E_{max}(t)$ for different values of t for Eq (4.3).

t	MADM-TR	MADM-WR
0	1.94×10^{-38}	5.55×10^{-38}
0.1	1.32×10^{-10}	5.29×10^{-15}
0.2	1.19×10^{-8}	2.22×10^{-14}

5. Conclusions

In this paper, we focused on studying the solution of Eq (1.1), which can be interpreted with different implications in mathematical physics and in contact problems where it can be defined as

$$\mu\phi(x,t) = f(x,t) + \lambda \int_0^t \int_0^1 F(t,\tau)K(x,y)G(y,\tau,\phi(y,\tau))dyd\tau \tag{5.1}$$

under the dynamic conditions

$$\int_0^1 \phi(x,t)dx = \mathcal{N}_1(t), \text{ and } \int_0^1 x\phi(x,t)dx = \mathcal{N}_2(t), \tag{5.2}$$

and where the following expression can be considered in the mathematical physics problems

$$f(x,t) = \frac{\pi}{\theta_1 + \theta_2}\left[\gamma(t) + \beta(t)x - h_1(x) - h_2(x)\right], \; x \in [0,1], \; t \in [0,T], T < 1$$
$$\text{and } \theta_i = \frac{1-\mu_i}{\pi E_i}, \quad i = 1,2.$$

Here, μ and λ were constants; however, they could be complex and their physical implications may vary. Moreover, μ_i is the Poisson's ration and E_i is the Young's coefficient of each material. IE (5.1) under conditions (5.2) was investigated throught the contact problem in the theory of elasticity of two rigid surfaces $G_i,\ i = 1, 2$ having two elastic materials occupying the contact domain [0, 1] where the two functions $h_i(x) \in L_2[0, 1]$ represent and describe the equations of the upper and lower surfaces. The upper surface was impressed by a given variable force in time $\mathcal{N}_1(t), 0 \le t \le T < 1$, with an eccentricity of application $e(t)$ and a given moment $\mathcal{N}_2(t)$ in consideration of the rigid displacements $\gamma(t)$ and $x\beta(t)$, respectively, through time $t \in [0, T]$ and position $x \in [0, 1]$. From the above discussions, the unknown function $\phi(x, t)$ represented the difference in the normal stresses between the two layers. Moreover, the kernel of position $K(x, y)$ depended on the properties of materials of the contact domain, whereas the known positive function $F(t, \tau)$ represented the characteristic function of the material resistance through time t with $F(0, 0) = \text{constant} \neq 0$.

Furthermore, the normality and continuity of NV-FIEs with phase lag in the space $L_2[a, b] \times C[0, T]$ were presented to investigate the uniqueness and existence of the solution using the Banach's fixed point theorem which is used in case of failure of Picard's method. Moreover, A new MADM based on quadrature rules, which is used in case the definite integral is extremely hard, was proposed to obtain the best approximate solutions of NV-FIEs with a phase lag. Illustrative plots of the method's applications were provided to prove the validity and accuracy of the proposed methods and to calculate the error for each method. Based on the results, the accuracy of MADM with quadrature formulas can be assigned in the order of MADM-Weddle's rule > MADM-Trapezoidal rule. Thus, compared to other rules, MADM-Weddle's rule, having the same relative accuracy of MADM, is the best approach to approximate the solution of NV-FIEs.

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Conflict of interest

The authors declare that they have no competing interests.

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